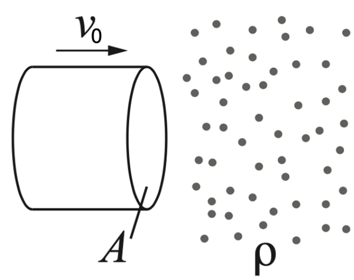
Physics Placement Exam: Classical Mechanics and Electromagnetism

Date goes here

Please do all six problems. Generous partial credit may be awarded to partial solutions provided your work is organized and legible. Note that various formulas that may be useful are on the last page of the exam.

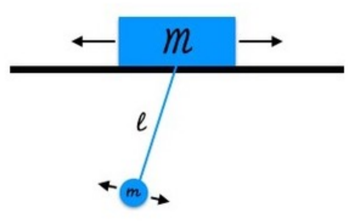
You must not use your phone, calculator, or any other device with messaging capabilities during the exam.

Problem 1: A spacecraft of mass m_0 is coasting with velocity v_0 when at $t = 0$ it encounters a stationary dust cloud of mass density ρ. Assume the spacecraft is a cylinder with cross-sectional area A, that v_0 is in the direction of the cylinder axis, and that the dust sticks to its front as it moves into the cloud.



a) Find an expression for the acceleration of the spacecraft
b) Find the speed and the mass of the spacecraft as a function of time.

Problem 2: A simple pendulum of mass m and length l is connected at its point of support to a block of mass M. The block is free to slide along a frictionless horizontal track.



a. Find the Lagrangian for this system assuming it is placed in a uniform gravitational field. Find Lagrange's equations.
b. What are the constants of motion?
c. What are the eigenfrequencies associated with small oscillations about the equilibrium position of this system? Describe qualitatively the motion of the system associated with each frequency. You can use diagrams to help explain.

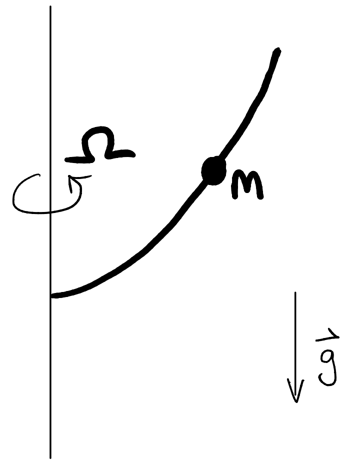


Figure 1: Setup for Problem 2.

Problem 3:

A particle of mass m is constrained to move along a frictionless wire. The wire is curved such that, in cylindrical coordinates, it follows $z(r) = z_0\ (r/a)^\beta$, where a and z_0 are positive constants with units of length and $\beta \geq 1$ is also constant. The wire is rotating at a constant angular velocity Ω around the vertical direction, as shown in Fig. 1. The system is in a uniform gravitational field with gravitational acceleration g pointing downward.

a) Write the Lagrangian for an unconstrained mass in a uniform gravitational field using a cylindrical coordinate system with the z-axis in the vertical direction.

b) Write explicitly the constraint equations that force the mass to remain on the rotating wire.

c) Determine the equations of motion for all the coordinates, introducing Lagrange multipliers for the forces of constraint.

d) Obtain expressions for the forces of constraint. Give a physical interpretation of these forces.

e) Construct the Hamiltonian. Is it equal to the total energy? Is the energy of the system conserved?

f) Under certain conditions the mass will remain stationary with respect to the wire. Find those conditions when $\beta = 1$ and provide a physical interpretation of your result.

Problem 4: The electric charge density is given by

$$\rho(\vec{r}) = Q_1 \delta^3(\vec{r}) + \frac{Q_2}{4\pi r^2}\delta(r - R) \quad .$$

That is, there is a point charge Q_1 at the origin and a surface charge $\sigma = \frac{Q_2}{4\pi R^2}$ on the surface of a sphere of radius R centered about the origin.
What is the electrostatic energy of this system of charges? (Show that your answer is correct in appropriate limits.)

Problem 5: Consider a spherical shell of radius R. Suppose $\nabla^2\Phi = 0$ both for $r < R$ and for $r > R$, and that the potential on the surface of the shell is given by $\Phi(r = R, \theta, \phi) = V_0\,(3\cos^2\theta - 1)$.
a) What is Φ when $r < R$?
b) What is Φ when $r > R$?
c) What is the surface charge on the spherical shell?

Problem 6: Consider an electromagnetic wave $\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$ propogating in the $\hat{z}$ direction, which is incident upon a free electron of mass m and charge q. Choose a coordinate system in which the unperturbed electron sits at the origin: x = y = z = 0.

a) Write down the equation of motion (Newton's 2nd law) for the displacement x of the electron due to the electric force of the passing wave. You may neglect radiative damping.

b) Assuming that the motion of the electron is harmonic: $x = x_0 e^{-i\omega t}$, find the amplitude of the electron displacement x_0.

c) What is the dipole moment of the electron induced by the passing EM wave? Assuming that electrons are distributed uniformly throughout space with number density n_e (thus forming an electron plasma), what is the resulting polarization $\vec{P}$ of the plasma?
(Recall that polarization in this context is the electric dipole moment per unit volume.)

d) Calculate the index of refraction n of the plasma using the relationship between the permittivity and the polarization $\epsilon = \epsilon_0 + \frac{P}{E}$. (you may assume $\mu = \mu_0$)

e) For what frequencies ω is n imaginary? Physically, what does an imaginary index of refraction mean in this context?

Potentially Useful Equations and Definitions

Spherical coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2} \quad , \quad L^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l,m}(\theta,\phi)\left[\Phi_{lm}^{(1)}\, r^l + \Phi_{lm}^{(2)}\,\frac{1}{r^{l+1}}\right] \quad \text{satisfies} \quad \nabla^2\Phi = 0 \quad .$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \quad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,\cos\theta$$

$$Y_{22}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}, \quad Y_{21}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta\cos\theta\, e^{i\phi}, \quad Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m\, Y_{lm}^*(\theta,\phi)$$

Electrostatic energy (W):

Discrete charges:

$$W = \frac{1}{8\pi\epsilon_0}\,\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}}$$

Continuous charge distribution:

$$W = \frac{\epsilon_0}{2}\,\int \mathbf{E}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\; d^3r = \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r})\; d^3r$$

Physics Placement Exam
Quantum Mechanics, Statistical Mechanics and Thermodynamics
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Problem 1: A beam of spin-$\frac{1}{2}$ particles each with charge $+e$ is directed along the $+x$ direction. Each particle in the beam is known to be in the $|+\mathbf{z}\rangle$ state. At time $t = 0$ the beam enters a uniform magnetic field B_0 in the x-z plane oriented at an angle θ with respect to the z-axis. At a later time $t = T$, the particles enter a Stern-Gerlach (SGy) device with magnetic field gradient directed along the y-direction.

What is the probability that the SG$\mathbf{y}$ device will find the particles to be in the $|+\mathbf{y}\rangle$ state?

Recall that $|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$, where θ and ϕ are the standard polar and azimuthal angles, respectively, in spherical coordinates of the vector $\mathbf{n}$.

Problem 2: Consider a simple harmonic oscillator with frequency ω and energy eigenfunctions $\psi_n(x)$.

a) Find the expectation value of x^2 in the n-th energy eigenstate.

b) Suppose that the oscillator is in a state such that a measurement of the energy would yield either $\frac{1}{2}\hbar\omega$ or $\frac{5}{2}\hbar\omega$, with equal probability. What is smallest possible value of $\langle x^2\rangle$ in such a state?

c) Suppose that the state in part (b) begins with the minimum value of $\langle x^2\rangle$ consistent with part a) at time $t = 0$. Determine $\langle x^2\rangle$ as a function of time.

Problem 3: A hydrogen atom is placed in electric field $\mathbf{E}(t)$ that is uniform in the z-direction and has the following time-dependence:

$$\begin{aligned} E_z(t) &= 0 && \text{for } t < 0 \\ E_z(t) &= E_0 e^{-\gamma t} && \text{for } t \geq 0 \end{aligned}$$

yielding, for $t \geq 0$, the following time-dependent potential energy:

$$V(t) = eE_0 e^{-\gamma t}\, r \cos\theta \quad .$$

If the hydrogen atom is initially in the ground state, what is the probability that it will be found in the 2P state as $t \to \infty$? Work to first order in E_0.

Problem 4: A flat box of height ℓ and width and depth L with $\ell \ll L$ contains a monoatomic ideal gas. You may assume the gas is arbitrarily dilute, but you should distinguish between low and high temperatures relative to the energy scale set by the quantum energy gap for momentum excitations in the vertical direction.

a) Compute the heat capacity C_V as a function of temperature, neglecting the role of Bose or Fermi statistics.

b) There is a regime where even for an arbitrarily dilute gas, you may no longer ignore the role of quantum statistics (assuming there is more than one particle in the box). Find the temperature where this becomes the case.

Problem 5: Consider a site which can be in one of three states: empty (energy 0), one electron of spin $+\frac{1}{2}$ (energy Δ), one electron of spin $-\frac{1}{2}$ (energy also Δ). States with two or more particles are forbidden. Suppose the site is coupled to a reservoir of chemical potential μ and temperature T.

a) Find the grand partition function.

b) Determine the chemical potential μ at which the expectation value of the number of particles on the site is $1/2$.

c) For this value of μ determine the average value of the magnetization $\bar{M} =< M >$ and the mean square fluctuation $< \left(M - \bar{M}\right)^2 >$. Assume that the magnetization in some units is +1 for the spin $+\frac{1}{2}$ state.

Problem 6: One mole of helium gas and one mole of nitrogen gas are initially kept in two separate containers of equal volume, both at room temperature. At some point you decide to connect the containers, allowing the gases to mix.

a) How much entropy was generated by mixing the gases?

b) How much energy will you need at least to reverse this process and restore the original state, assuming you operate at room temperature?

(Provide numerical answers to both a) and b)), using $k_B = 1.381 \times 10^{-23}$ J/K .)

Potentially Useful Equations and Definitions

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{a} = \frac{1}{\sqrt{2}} \left[\frac{\hat{x}}{x_0} + i \frac{\hat{p}}{(\hbar/x_0)} \right] \quad , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left[\frac{\hat{x}}{x_0} - i \frac{\hat{p}}{(\hbar/x_0)} \right] \quad , \quad x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$$

$$\hat{a}\ \psi_n = \sqrt{n}\ \psi_{n-1} \quad , \quad \hat{a}^\dagger\ \psi_n = \sqrt{n+1}\ \psi_{n+1}$$

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi} x_0}} \frac{1}{\sqrt{2^n n!}}\ H_n(\xi) e^{-\xi^2/2} \quad , \quad \xi \equiv \frac{x}{x_0}$$

$$H_0(\xi) = 1 \quad , \quad H_1(\xi) = 2\xi \quad , \quad H_2(\xi) = 4\xi^2 - 2$$

$$\int_{-\infty}^{\infty} du\ e^{-\lambda u^2} = \sqrt{\frac{\pi}{\lambda}} \quad , \quad \int_{-\infty}^{\infty} du\ u^2\ e^{-\lambda u^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} \quad , \quad \int_{-\infty}^{\infty} du\ u^4\ e^{-\lambda u^2} = \frac{3}{4}\sqrt{\frac{\pi}{\lambda^5}}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta\ e^{i\phi}, \quad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\ \cos\theta$$

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$$Y_{l,-m}(\theta,\phi) = (-1)^m\ Y_{lm}^*(\theta,\phi)$$

For hydrogen atom,

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\ e^{-r/a_0} \quad , \quad R_{20}(r) = \frac{1}{\sqrt{2} a_0^{3/2}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \quad , \quad R_{21}(r) = \frac{1}{\sqrt{24} a_0^{3/2}}\ \frac{r}{a_0}\ e^{-r/2a_0}$$

$$a_0 = \frac{\hbar^2}{me^2} \quad , \quad E_n = -\frac{me^4}{2\hbar^2 n^2}$$

First-order time-dependent perturbation theory

$$c_{f\leftarrow i}(t) = \delta_{fi} - \frac{i}{\hbar} \int_0^t e^{i(E_f - E_i)t'/\hbar} \langle E_f | \hat{H}_1(t') | E_i \rangle\ dt'$$

where the energies are those of the unperturbed eigenstates and $\hat{H}_1(t)$ is a time-dependent perturbation.